

REMARKS ON THE ORBITAL STABILITY OF GROUND STATE SOLUTIONS OF FKDV AND RELATED EQUATIONS

FELIPE LINARES, DIDIER PILOD, AND JEAN-CLAUDE SAUT

ABSTRACT. The aim of this paper is to provide a proof of the (conditional) orbital stability of solitary waves solutions to the fractional Korteweg- de Vries equation (fKdV) and to the fractional Benjamin-Bona-Mahony (fBBM) equation in the L^2 subcritical case. We also discuss instability and its possible scenarios.

1. INTRODUCTION

This paper continues the study initiated in [38] of the fractional Korteweg-de Vries equation (fKdV)

$$(1.1) \quad u_t + uu_x - D^\alpha u_x = 0, \quad u(\cdot, 0) = u_0,$$

and of its Benjamin-Bona-Mahony counterpart (fBBM)

$$(1.2) \quad u_t + u_x + uu_x + D^\alpha u_t = 0,$$

where $D^\alpha = (-\partial_x^2)^{\frac{\alpha}{2}}$ and $0 < \alpha < 1$. D^α is defined via Fourier transform by

$$(D^\alpha f)^\wedge(\xi) = |\xi|^\alpha \hat{f}(\xi).$$

The fKdV equation is a toy model to understand the interaction between nonlinearity and dispersion. The choice is here to fix the quadratic nonlinearity which appears “generically” in many physical contexts and to vary (lower) the dispersion (see [38, 34]).

Equations like (1.1) but with an inhomogeneous symbol can be derived rigorously as water waves models (in the small amplitude, long wave regime) [36, 37]. For instance the so-called Whitham equation [59] is of fKdV type with a weak dispersion, that is

$$(1.3) \quad u_t + uu_x + \int_{-\infty}^{\infty} k(x-y)u_x(y,t)dy = 0.$$

This equation can also be written on the form

$$(1.4) \quad u_t + uu_x - Lu_x = 0,$$

where the Fourier multiplier operator L is defined by

$$\widehat{L}f(\xi) = p(\xi)\hat{f}(\xi),$$

with $p = \hat{k}$. In the original Whitham equation, the kernel k was given by

$$(1.5) \quad k(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\tanh \xi}{\xi} \right)^{\frac{1}{2}} e^{ix\xi} d\xi,$$

that is $p(\xi) = \left(\frac{\tanh \xi}{\xi} \right)^{\frac{1}{2}}$ which behaves like $|\xi|^{-\frac{1}{2}}$ for large frequencies and like $1 - \frac{\xi^2}{6}$ for small frequencies.

When surface tension is included the symbol p above has to be changed to $p_S(\xi) = (1 + \beta|\xi|^2)^{\frac{1}{2}} \left(\frac{\tanh \xi}{\xi} \right)^{\frac{1}{2}}$, where $\beta \geq 0$ measures the surface tension effects. This leads to the *extended Whitham equation* where the symbol $p_S(\xi)$ behaves as $\beta|\xi|^{\frac{1}{2}}$ for large frequencies and as $1 - (\frac{1}{6} - \beta)\xi^2$ for small frequencies.

The equation (1.1) is invariant under the scaling transformation

$$u_\lambda(x, t) = \lambda^\alpha u(\lambda x, \lambda^{\alpha+1} t),$$

for any positive number λ . A straightforward computation shows that $\|u_\lambda\|_{\dot{H}^s} = \lambda^{s+\alpha-\frac{1}{2}} \|u\|_{\dot{H}^s}$, in particular the value $\alpha = \frac{1}{2}$ corresponds to the L^2 critical case.

One associates to (1.1), (1.2) the energy space $H^{\frac{\alpha}{2}}(\mathbb{R})$, motivated by their conservation laws. The following quantities are formally conserved by the flow associated to (1.1),

$$(1.6) \quad M(u) = \frac{1}{2} \int_{\mathbb{R}} u^2(x, t) dx,$$

and

$$(1.7) \quad E(u) = \int_{\mathbb{R}} \left(\frac{1}{2} |D^{\frac{\alpha}{2}} u(x, t)|^2 - \frac{1}{6} u^3(x, t) \right) dx.$$

Note that by the Sobolev embedding $H^{\frac{1}{6}}(\mathbb{R}) \hookrightarrow L^3(\mathbb{R})$, $H(u)$ is well-defined if and only if $\alpha \geq \frac{1}{3}$, in other words $\alpha = \frac{1}{3}$ is the energy critical exponent.

On the other hand, there is no energy critical exponent α in the case of the fBBM equation (1.2) since the momentum

$$N(u) = \frac{1}{2} \int_{\mathbb{R}} (u^2 + |D^{\frac{\alpha}{2}} u|^2) dx$$

makes obviously always sense for $u \in H^{\frac{\alpha}{2}}(\mathbb{R})$. Another conserved quantity for (1.2) is the Hamiltonian

$$F(u) = \int_{\mathbb{R}} \left(\frac{u^2}{2} + \frac{u^3}{6} \right),$$

which makes sense when $u \in H^{\frac{\alpha}{2}}(\mathbb{R})$, $\alpha \geq \frac{1}{3}$.

There is apparently no published result on the orbital stability for solitary waves of fractional KdV equations (fKdV) (1.1) or fractional BBM equations

(1.2) in the range $0 < \alpha < 1$. The known existence proofs (see [24, 23] and also [2]) use M. Weinstein's argument, looking for the best constant in the fractional Gagliardo-Nirenberg inequality

$$(1.8) \quad \int_{\mathbb{R}} |u|^3 dx \leq C \left(\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} u|^2 dx \right)^{\frac{1}{2\alpha}} \left(\int_{\mathbb{R}} u^2 dx \right)^{\frac{3\alpha-1}{2\alpha}}.$$

This gives the existence in the energy sub-critical case $\alpha > \frac{1}{3}$, but of course not any kind of stability, which should be true only in the L^2 subcritical case, $\alpha > \frac{1}{2}$.

Orbital stability issues for the fractional Schrödinger equations has been considered in [16]. We will restrict to the fKdV equation (1.1) with *homogeneous* dispersion, (but the method extends obviously to the non homogeneous case).

The solitary waves are solutions of (1.1) of the form $u(x, t) = Q_c(x - ct)$, $c > 0$ where Q_c belongs to the energy space $H^{\frac{\alpha}{2}}(\mathbb{R})$ and they should thus satisfy the equation

$$(1.9) \quad D^\alpha Q_c + cQ_c - \frac{1}{2}Q_c^2 = 0.$$

The energy identity

$$(1.10) \quad \int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 dx + c \int_{\mathbb{R}} Q_c^2 dx - \frac{1}{2} \int_{\mathbb{R}} Q_c^3 dx = 0$$

and the Pohojaev identity

$$(1.11) \quad \frac{\alpha-1}{2} \int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 dx - \frac{c}{2} \int_{\mathbb{R}} Q_c^2 dx + \frac{1}{6} \int_{\mathbb{R}} Q_c^3 dx = 0$$

which in turn is a consequence of the identity (see for instance Lemma 3 in [31])

$$(1.12) \quad \int_{\mathbb{R}} (D^\alpha \phi) x \phi' dx = \frac{\alpha-1}{2} \int_{\mathbb{R}} |D^{\frac{\alpha}{2}} \phi|^2 dx,$$

imply

$$(1.13) \quad (3\alpha-1) \int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 dx - c \int_{\mathbb{R}} Q_c^2 dx = 0$$

proving that no finite energy solitary waves exist in the energy subcritical case $\alpha > 1/3$ when $c \leq 0$ (see [38]).

J. Albert has considered in [1] the case $\alpha \geq 1$, for (1.1) so we will focus on the case $1/2 < \alpha < 1$, which is L^2 sub-critical for (1.1). In his notation, $s = \alpha/2$. The proof in [1] is inspired by an old idea of Boussinesq, revisited by Benjamin [9] (and by Cazenave-Lions [14] for NLS type equations) and consists in using the concentration-compactness method of P.-L. Lions to prove the existence of a minimizer of the Hamiltonian (energy) with fixed momentum (L^2 norm). The proof gives nearly for free the orbital stability

of the set of minimizers, assuming that the corresponding Cauchy problem is globally well-posed in the energy space, at least for initial data close to a solitary wave (a fact which is conjectured but still unproved in the case of fKdV when $1/2 < \alpha < 1$.)

Uniqueness and positivity properties of a class of solitary waves (ground states) have been investigated in [21, 22, 24, 23] among others. We recall that existence of solitary waves of arbitrary positive velocities has been established in the *energy subcritical case*, that is when $\alpha > \frac{1}{3}$ while no localized solitary waves exist when $0 < \alpha < \frac{1}{3}$ (see the argument above), that is in the energy supercritical case. It is worth noticing that existence of solitary waves for the original Whitham equation has been established in [17] by exploiting that the dispersion approaches that of the KdV equation in the long wave limit.

On the other hand, numerical simulations ([34]) suggest that the Cauchy problem for (1.1) is globally well-posed for $\alpha > \frac{1}{2}$, a typical solution decomposing into solitary waves plus radiation, which would give a positive answer to the *soliton resolution conjecture* ([57]). One aim of this note is to provide a (small) step towards this conjecture, namely to prove that the solitary waves are orbitally stable for this range of α 's.¹

The paper is organized as follows. In the following section we consider the fKdV equation. The next section deals with the fBBM equation. Lastly we initiate an extension to fractional Kadomtsev-Petviashvili I (fKPI) equations.

Notations. We will denote $|\cdot|_p$ the norm in the Lebesgue space $L^p(\mathbb{R})$, $1 \leq p \leq \infty$ and $\|\cdot\|_s$ the norm in the Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$. We will denote \hat{f} or $\mathcal{F}(f)$ the Fourier transform of a tempered distribution f . For any $s \in \mathbb{R}$, we define $D^s f$ by its Fourier transform $\widehat{D^s f}(\xi) = |\xi|^s \hat{f}(\xi)$.

2. THE FKDV

We will follow closely the strategy of [1], which was used to prove the orbital stability of the KdV solitary waves and the (conditional) orbital stability for (1.1) in the case $\alpha \geq 1$ and related equations. We just indicate the differences. We will assume in this section that $\frac{1}{2} < \alpha < 1$. We recall that

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} [|D^{\frac{\alpha}{2}} u|^2 - \frac{1}{3} u^3] dx \quad \text{and} \quad M(u) = \frac{1}{2} \int_{\mathbb{R}} u^2 dx.$$

For $q > 0$ fixed, we set

$$(2.1) \quad I_q = \inf_{u \in H^{\frac{\alpha}{2}}(\mathbb{R})} \{E(u) : M(u) = q\}.$$

We will denote by G_q the set (possibly empty) of minimizers.

Lemma 2.1. *For any $q > 0$ one has $-\infty < I_q < 0$.*

¹Actually we prove a *conditional* stability result since we do not know that the solutions of the Cauchy problem are global in this case.

Proof. By Sobolev and a standard interpolation inequality, one has for any $\epsilon > 0$ and $v \in H^{\frac{\alpha}{2}}(\mathbb{R})$ such that $M(v) = q$,

$$\left| \int_{\mathbb{R}} v^3 dx \right| \leq \|v\|_{\frac{1}{6}}^3 \leq C \|v\|_0^{\frac{3\alpha-1}{\alpha}} \|v\|_{\frac{\alpha}{2}}^{\frac{1}{\alpha}} \leq \epsilon \|v\|_{\frac{\alpha}{2}}^2 + C_{\epsilon} \|v\|_0^{\frac{2(3\alpha-1)}{2\alpha-1}}.$$

Now we write as in the proof of Lemma 3.4 in [1]

$$\begin{aligned} E(v) &= E(v) + M(v) - M(v) \\ &= \frac{1}{2} \int_{\mathbb{R}} [|D^{\frac{\alpha}{2}} v|^2 + v^2] dx - \frac{1}{6} \int_{\mathbb{R}} v^3 dx - M(v) \\ (2.2) \quad &\geq \left(\frac{1}{2} - \frac{\epsilon}{6} \right) \|v\|_{\frac{\alpha}{2}}^2 - q - C'_{\epsilon} q^{(3\alpha-1)/(2\alpha-1)} \\ &\geq -q - C'_{\epsilon} q^{(3\alpha-1)/(2\alpha-1)} > -\infty. \end{aligned}$$

The fact that $I_q < 0$ is easily checked by scaling as in the proof of Lemma 3.4 in [1]. \square

So I_q exists and is finite, and the concentration-compactness method is used to prove that it is achieved. A first step is to prove that the minimizing sequences are bounded.

Lemma 2.2. *If $\{v_n\}$ is a minimizing sequence for I_q then there exist positive constants C and δ such that*

1. $\|v_n\|_{\frac{\alpha}{2}} \leq C$ for all n and
2. $|v_n|_3 \geq \delta$ for all n sufficiently large.

Proof. Let $\{v_n\}$ be a minimizing sequence for I_q . Firstly, one has by a previous estimate

$$\frac{1}{2} \|v_n\|_{\frac{\alpha}{2}}^2 = E(v_n) + \frac{1}{2} \int_{\mathbb{R}} v_n^2 dx + \frac{1}{6} \int_{\mathbb{R}} v_n^3 dx \leq |E(v_n)| + \epsilon \|v_n\|_{\frac{\alpha}{2}}^2 + C(q),$$

proving 1.

In order to prove 2, we argue by contradiction, assuming that for any $k \in \mathbb{N}$ there exists a subsequence v_{n_k} such that $|v_{n_k}|_3 \leq 1/k, \forall k$. This implies

$$I_q = \lim_{k \rightarrow \infty} \left(\frac{1}{2} \int_{\mathbb{R}} |D^{\frac{\alpha}{2}} v_{n_k}|^2 - \frac{1}{6} \int_{\mathbb{R}} v_{n_k}^3 dx \right) \geq - \lim_{k \rightarrow \infty} \frac{1}{6} \int_{\mathbb{R}} v_{n_k}^3 dx = 0,$$

in contradiction with Lemma 2.1. \square

The next step is to prove the sub-additivity of I_q , (see [40, 41]).

Lemma 2.3. *For all $q_1, q_2 > 0$, one has*

$$I_{q_1+q_2} < I_{q_1} + I_{q_2}.$$

Proof. As in Lemma 2.4 in [1] the proof follows from a homogeneity argument which we give by sake of completeness. For all $\theta > 0$ and $q > 0$ we claim that

$$(2.3) \quad I_{\theta q} = \theta^{(3\alpha-1)/(2\alpha-1)} I_q.$$

To prove the claim, we set for any function $v \in H^{\frac{\alpha}{2}}(\mathbb{R})$,

$$v_\theta(x) = \theta^{\alpha/(2\alpha-1)} v(\theta^{1/(2\alpha-1)} x).$$

Then

$$M(v_\theta) = \theta M(v),$$

and

$$E(v_\theta) = \theta^{(3\alpha-1)/(2\alpha-1)} E(v).$$

Hence

$$\begin{aligned} I_{\theta q} &= \inf\{E(v_\theta) : M(v_\theta) = \theta q\} \\ &= \inf\{E(v_\theta) : M(v) = q\} \\ (2.4) \quad &= \inf\{\theta^{(3\alpha-1)/(2\alpha-1)} E(v) : M(v) = q\} \\ &= \theta^{(3\alpha-1)/(2\alpha-1)} I_q. \end{aligned}$$

It follows then (by choosing $q = 1$ and $\theta = q_1 + q_2$ in (2.3)) that

$$\begin{aligned} I_{q_1+q_2} &= (q_1 + q_2)^{(3\alpha-1)/(2\alpha-1)} I_1 \\ &< \left(q_1^{(3\alpha-1)/(2\alpha-1)} + q_2^{(3\alpha-1)/(2\alpha-1)} \right) I_1 = I_{q_1} + I_{q_2}. \end{aligned}$$

□

As usual in the concentration compactness method, we associate to any minimizing sequence $\{v_n\}$ the sequence of nondecreasing functions $\mathfrak{M}_n : [0, \infty) \rightarrow [0, q]$ defined by

$$\mathfrak{M}_n(r) = \sup_{y \in \mathbb{R}} \int_{y-r}^{y+r} |v_n|^2 dx.$$

By an elementary argument, $\{\mathfrak{M}_n\}$ has a subsequence, still denoted by $\{\mathfrak{M}_n\}$, which converges uniformly on compact sets to a nondecreasing function $\mathfrak{M} : [0, \infty) \rightarrow [0, q]$. Let

$$\lambda = \lim_{r \rightarrow \infty} \mathfrak{M}(r), \quad \text{so that } 0 \leq \lambda \leq q.$$

We will examine successively the three (mutually exclusive) possibilities, $\lambda = q$ (compactness), $\lambda = 0$ (vanishing), $0 < \lambda < q$ (dichotomy).

The compactness case is the good one in virtue of the following lemma.

Lemma 2.4. *Assume that $\lambda = q$. Then there exists a sequence of real numbers $\{y_n\}_{n \in \mathbb{N}}$ such that*

1. *For every $z < q$ there exists $r = r(z)$ such that*

$$\int_{y_n-r}^{y_n+r} |v_n|^2 > z$$

for all sufficiently large n .

2. *The sequence $\{\tilde{v}_n\}$ defined by*

$$\tilde{v}_n(x) = v_n(x + y_n) \quad \text{for all } x \in \mathbb{R}$$

has a subsequence which converges in $H^{\frac{\alpha}{2}}(\mathbb{R})$ to a function $g \in G_q$. In particular, G_q is not empty.

Proof. The proof is classical and follows exactly that of Lemma 2.5 in [1], replacing $H^1(\mathbb{R})$ by $H^{\frac{\alpha}{2}}(\mathbb{R})$. \square

The next technical lemma will be use to prove that vanishing does not occur.

Lemma 2.5. *Suppose that $B > 0$ and $\delta > 0$ are given. Then there exists $\eta = \eta(B, \delta)$ such that if $v \in H^{\frac{\alpha}{2}}(\mathbb{R})$, $\alpha > 1/3$, with $\|v\|_{\frac{\alpha}{2}} \leq B$ and $|v|_3 \geq \delta$, then*

$$\sup_{y \in \mathbb{R}} \int_{y-2}^{y+2} |v(x)|^3 dx \geq \eta.$$

Proof. The proof follows exactly that of Lemmas 3.7, 3.8, 3.9 in [2] (see also Lemma 3.3 in [1]) in the case $\alpha = 1$. \square

The following key lemma shows that dichotomy occurs when $0 < \lambda < q$.

Lemma 2.6. *We still consider a minimizing sequence $\{v_n\}$. Then for every $\epsilon > 0$ there exist $N \in \mathbb{N}$ and sequences $\{g_N, g_{N+1}, \dots\}$ and $\{h_N, h_{N+1}, \dots\}$ of functions in $H^{\frac{\alpha}{2}}(\mathbb{R})$ such that for every $n \geq N$,*

1. $|M(g_n) - \lambda| < \epsilon$
2. $|M(h_n) - (q - \lambda)| < \epsilon$
3. $E(v_n) \geq E(g_n) + E(h_n) - \epsilon$.

Proof. Statements 1 and 2 are pretty general and a proof can be found for instance in that of Lemma 2.6 in [1] (see also a sketch of the proof below). Statement 3 is more delicate because of the non locality of D^α .

To prove 3, we follow closely the proof of Lemmas 2.6 and 3.8 in [1]. Let $\phi \in C_0^\infty[-2, 2]$ be such that $\phi \equiv 1$ on $[-1, 1]$, and let $\psi \in C^\infty(\mathbb{R})$ be such that $\phi^2 + \psi^2 \equiv 1$ on \mathbb{R} . For each $r \in \mathbb{R}$, define $\phi_r(x) = \phi(x/r)$ and $\psi_r(x) = \psi(x/r)$. Coming back to the definition of \mathfrak{M} , $\epsilon > 0$ being fixed, for all sufficiently large values of r one has

$$\lambda - \epsilon < \mathfrak{M}(r) \leq \mathfrak{M}(2r) \leq \lambda.$$

Such a value of r being fixed, one can choose N so large that

$$\lambda - \epsilon < \mathfrak{M}_n(r) \leq \mathfrak{M}_n(2r) \leq \lambda + \epsilon$$

for all $n \geq N$. Hence for each $n \geq N$, one can find y_n such that

$$(2.5) \quad \int_{y_n-r}^{y_n+r} |v_n|^2 dx > \lambda - \epsilon$$

and

$$(2.6) \quad \int_{y_n-2r}^{y_n+2r} |v_n|^2 dx < \lambda + \epsilon$$

Define $g_n(x) = \phi_r(x - y_n)v_n(x)$ and $h_n(x) = \psi_r(x - y_n)v_n(x)$. Clearly g_n and h_n satisfy statements 1 and 2.

We now write

$$\begin{aligned}
(2.7) \quad E(g_n) + E(h_n) &= \frac{1}{2} \left[\int \phi_r^2 v_n D^\alpha v_n dx + \int \phi_r v_n [D^\alpha, \phi_r] v_n dx \right] \\
&\quad + \frac{1}{2} \left[\int \psi_r^2 v_n D^\alpha v_n dx + \int \psi_r v_n [D^\alpha, \psi_r] v_n dx \right] \\
&\quad - \frac{1}{6} \int \phi_r^2 v_n^3 dx - \frac{1}{6} \int \psi_r^2 v_n^3 dx \\
&\quad + \frac{1}{6} \int (\phi_r^2 - \phi_r^3) v_n^3 dx + \frac{1}{6} \int (\psi_r^2 - \psi_r^3) v_n^3 dx \\
&= E(v_n) + \int \phi_r v_n [D^\alpha, \phi_r] v_n dx + \int \psi_r v_n [D^\alpha, \psi_r] v_n dx \\
&\quad + \frac{1}{6} \int (\phi_r^2 - \phi_r^3) v_n^3 dx + \frac{1}{6} \int (\psi_r^2 - \psi_r^3) v_n^3 dx,
\end{aligned}$$

where we have used that $\phi^2 + \psi^2 \equiv 1$.

As in [1] we want to prove that the sum of the two commutators is $O(1/r^\beta)$ for some $\beta > 0$ and that the sum of the two other terms is $O(\epsilon)$. For the later this is exactly as in [1]. For the commutator, since in his case $\alpha = 1$, Albert uses that $||[D, \theta]f|_2 \leq C|\theta'|_\infty |f|_2$ and this is fine since $|\phi'_r|_\infty = 1/r|\phi'|_\infty$.

For $\alpha < 1$, we will use instead the fractional Leibniz rule of Kenig, Ponce and Vega (*cf* Theorem A.8 and A.12 in the appendix of [32]).

Lemma 2.7 (Fractional Leibniz Rule). *Let $0 < \alpha < 1$, $1 < p$, $p_1, p_2 < +\infty$ and $\alpha_1, \alpha_2 \in [0, \alpha]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\alpha = \alpha_1 + \alpha_2$. Then*

$$(2.8) \quad |D^\alpha(fg) - fD^\alpha g - gD^\alpha f|_p \lesssim |D^{\alpha_1} f|_{p_1} |D^{\alpha_2} g|_{p_2}.$$

Moreover if $\alpha_1 = 0$, then $p_1 = +\infty$ is allowed.

First, we estimate $||[D^\alpha, \phi_r]v_n|_2$. Observe that

$$(2.9) \quad |[D^\alpha, \phi_r]v_n|_2 \leq |D^\alpha(\phi_r v_n) - \phi_r D^\alpha v_n - v_n D^\alpha \phi_r|_2 + |v_n D^\alpha \phi_r|_2$$

Thus, by using (2.8) with $f = v_n$, $g = \phi_r$, $p = 2$, $p_1 = p_2 = 4$ and $\alpha_2 = \alpha$, $\alpha_1 = 0$, we get that

$$(2.10) \quad |[D^\alpha, \phi_r]v_n|_2 \lesssim |v_n|_4 |D^\alpha(\phi_r)|_4.$$

On the one hand due to the Sobolev embedding $H^{\frac{1}{4}}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$ and the fact that $\{v_n\}$ is bounded in $H^{\frac{\alpha}{2}}(\mathbb{R})$ with $\frac{\alpha}{2} > \frac{1}{4}$, there exists $C > 0$ such that

$$(2.11) \quad |v_n|_4 \leq C.$$

On the other hand, a direct computation yields

$$(2.12) \quad |D^\alpha(\phi_r)|_4 = r^{\frac{1}{4}-\alpha} |D^\alpha \phi|_4 = \mathcal{O}(r^{\frac{1}{4}-\alpha}),$$

since $\phi \in C_0^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$. Thus, we conclude gathering (2.10)–(2.12) that

$$(2.13) \quad |[D^\alpha, \phi_r]v_n|_2 = \mathcal{O}(r^{\frac{1}{4}-\alpha}),$$

which is fine since $\alpha > \frac{1}{2}$.

We use the same strategy to estimate $||D^\alpha, \psi_r]v_n|_2$. From the definition of ψ , we have $\phi^2 + \psi^2 = 1$, so that we can write

$$\psi = 1 - \chi \quad \text{where} \quad \chi = 1 - \sqrt{1 - \phi^2} \in C_0^\infty(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}).$$

Moreover, it holds that $(D^\alpha(1))^\wedge(\xi) = c|\xi|^\alpha \delta_0 = 0$ in $\mathcal{S}'(\mathbb{R})$. Then

$$|D^\alpha(\psi_r)|_4 = |D^\alpha(\chi_r)|_4 = r^{\frac{1}{4}-\alpha} |D^\alpha \chi|_4 = \mathcal{O}(r^{\frac{1}{4}-\alpha}).$$

Therefore, we conclude arguing as above that

$$(2.14) \quad ||[D^\alpha, \psi_r]v_n|_2 = \mathcal{O}(r^{\frac{1}{4}-\alpha}).$$

Finally we have established that

$$E(g_n) + E(h_n) = E(v_n) + \mathcal{O}(r^{\frac{1}{4}-\alpha}) + \mathcal{O}(\epsilon),$$

which achieves the proof of 3. \square

As in [1] Corollary 2.7, one deduces from Lemma 2.6

Corollary 2.8. *If $0 < \lambda < q$, then*

$$I_q \geq I_\lambda + I_{q-\lambda}.$$

Corollary 2.8 shows why dichotomy cannot hold. We now prove that vanishing does not occur.

Lemma 2.9. *For every minimizing sequence, $\lambda > 0$.*

Proof. By Lemmas 2.2 and 2.5 there exist $\eta > 0$ and a sequence $\{y_n\}$ such that

$$\int_{y_n-2}^{y_n+2} |v_n|^3 dx \geq \eta \quad \text{for all } n.$$

Hence,

$$(2.15) \quad \begin{aligned} \eta &\leq \left(\int_{y_n-2}^{y_n+2} |v_n|^2 dx \right)^{1/2} \left(\int_{y_n-2}^{y_n+2} |v_n|^4 dx \right)^{1/2} \\ &\leq \left(\int_{y_n-2}^{y_n+2} |v_n|^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} |v_n|^4 dx \right)^{1/2} \\ &\leq C \left(\int_{y_n-2}^{y_n+2} |v_n|^2 dx \right)^{1/2}, \end{aligned}$$

where we have used the embedding $H^{\frac{\alpha}{2}}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R})$ when $\alpha \geq \frac{1}{2}$.

Thus

$$\lambda = \lim_{r \rightarrow \infty} M(r) \geq M(2) = \lim_{n \rightarrow \infty} M_n(2) \geq \frac{\eta}{C} > 0.$$

\square

We can now state and prove our main result. We first recall (see [38]) that the Cauchy problem for (1.1) is locally well-posed in $H^s(\mathbb{R})$, $s > s_\alpha = \frac{3}{2} - \frac{3\alpha}{8}$ in the sense that for any $u_0 \in H^s(\mathbb{R})$ with s as above, there exists a maximal time of existence $T_s \in (0, +\infty]$ and a unique solution u to (1.1) such that $u \in C([0, T_s]; H^s(\mathbb{R}))$ satisfying $Q(u(\cdot, t)) = Q(u_0)$ and $E(u(\cdot, t)) = E(u_0)$, $t \in [0, T_s)$.

Theorem 2.10. *Let $\frac{1}{2} < \alpha < 1$.*

1. *For every $q > 0$ there exists a nonempty set G_q of minimizers of (3.2) consisting of solitary waves of (1.1) with positive velocity. Moreover, if $\{v_n\}$ is a minimizing sequence for I_q , then the following assertions are true.*

2. *There exist a sequence $\{y_1, y_2, \dots\}$ and an element $g \in G_q$ such that $\{v_n(\cdot + y_n)\}$ has a subsequence converging strongly in $H^{\frac{\alpha}{2}}(\mathbb{R})$ to g .*

3.

$$\lim_{n \rightarrow \infty} \inf_{g \in G_q, y \in \mathbb{R}} \|v_n(\cdot + y) - g\|_{\frac{\alpha}{2}} = 0.$$

4.

$$\lim_{n \rightarrow \infty} \inf_{g \in G_q} \|v_n - g\|_{\frac{\alpha}{2}} = 0.$$

5. *The set G_q is stable in the following sense. For any $\epsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^s(\mathbb{R})$, $s > s_\alpha$, with*

$$\inf_{g \in G_q} \|u_0 - g\|_{\frac{\alpha}{2}} < \delta,$$

then the corresponding solution u emanating from u_0 of (1.1) satisfies

$$\inf_{g \in G_q} \|u(\cdot, t) - g\|_{\frac{\alpha}{2}} < \epsilon, \quad \forall 0 < t < T_s.$$

Proof. The proof is a classical application of the concentration-compactness method. By Lemmas 2.3, 2.6, 2.9 and Corollary 2.8 we deduce that $\lambda = q$.

We prove 2 by contradiction, assuming that there exist a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and $\epsilon > 0$ such that

$$\inf_{g \in G_q, y \in \mathbb{R}} \|v_{n_k}(\cdot + y) - g\|_{\frac{\alpha}{2}} \geq \epsilon$$

for all $k \in \mathbb{N}$. Since $\{v_{n_k}\}$ is also a minimizing sequence for I_q , statement 1 implies that there exist a sequence $\{y_k\}$ and $g_0 \in G_q$ such that

$$\liminf_{k \rightarrow \infty} \|v_{n_k}(\cdot + y_k) - g_0\|_{\frac{\alpha}{2}} = 0,$$

and this contradiction proves 2.

The stability statement 5 is classically proven by contradiction from 4. \square

We now relate the set G_q of minimizers to I_q to the *ground states* as defined in [24], Definition 2.1.

Definition 2.11. [24]

Let $\alpha > \frac{1}{3}$. A *ground state solution* of

$$(2.16) \quad D^{\frac{\alpha}{2}}Q + Q - Q^2 = 0,$$

is a positive and even solution that solves the minimization problem

$$(2.17) \quad J^\alpha(Q) = \inf \{ J^\alpha(u) : u \in H^{\frac{\alpha}{2}}(\mathbb{R}) \setminus \{0\} \},$$

where J^α is the Weinstein functional defined by

$$(2.18) \quad J^\alpha(u) = \left(\int_{\mathbb{R}} |u|^3 dx \right)^{-1} \left(\int_{\mathbb{R}} |D^{\frac{\alpha}{2}} u|^2 dx \right)^{\frac{1}{2\alpha}} \left(\int_{\mathbb{R}} u^2 dx \right)^{\frac{3\alpha-1}{2\alpha}}.$$

Lemma 2.12. *Let $q > 0$ and $\frac{1}{2} < \alpha < 1$. Any minimizer ψ of I_q writes*

$$(2.19) \quad \psi = cQ(c^{\frac{1}{\alpha}}(\cdot + y))$$

for some $y \in \mathbb{R}$ and $c > 0$ chosen to ensure that $\frac{1}{2} \int_{\mathbb{R}} \psi^2 dx = q$ holds and Q is a ground state solution of (2.16).

In order to prove Lemma 2.12, we recall the fundamental result² of Frank and Lenzmann in Theorem 2.4 of [24].

Theorem 2.13. *Let $\alpha > \frac{1}{3}$. Then, the ground state solution $Q = Q(|x|) > 0$ of equation (2.16) is unique.*

Furthermore, every minimizer $v \in H^{\frac{\alpha}{2}}(\mathbb{R})$ for the Weinstein functional J^α defined in (2.18) is of the form $v = \beta Q(\lambda(\cdot + y))$ for some $\beta \in \mathbb{C}$, $\beta \neq 0$, $\lambda > 0$ and $y \in \mathbb{R}$.

Proof of Lemma 2.12. Assume that $q > 0$ is fixed. Let Q be a ground state of (2.16) defined as above. Observe that for any $c > 0$, $Q_c = cQ(c^{\frac{1}{\alpha}}\cdot)$ is a solution to (1.9). It follows from (1.10) and (1.11) that

$$(2.20) \quad \int_{\mathbb{R}} |D^{\frac{\alpha}{2}} Q_c|^2 dx = \frac{c}{3\alpha - 1} \int_{\mathbb{R}} Q_c^2 dx$$

and

$$(2.21) \quad \int_{\mathbb{R}} Q_c^3 dx = \frac{6\alpha c}{3\alpha - 1} \int_{\mathbb{R}} Q_c^2 dx.$$

Therefore, a straightforward computation gives that

$$(2.22) \quad J^\alpha(Q_c) = \frac{(3\alpha - 1)^{1-\frac{1}{2\alpha}}}{6\alpha} c^{\frac{1}{2\alpha}-1} \|Q_c\|_{L^2} = \frac{(3\alpha - 1)^{1-\frac{1}{2\alpha}}}{6\alpha} \|Q\|_{L^2}.$$

Note in particular that the minimum of J^α is attained for every Q_c with $c > 0$. Moreover, we choose $c_\star > 0$ such that

$$(2.23) \quad M(Q_{c_\star}) = q \quad \Leftrightarrow \quad c_\star = \left(\frac{2q}{\|Q\|_{L^2}^2} \right)^{\frac{\alpha}{2\alpha-1}}.$$

Another easy computation yields

$$(2.24) \quad E(Q_{c_\star}) = \frac{c_\star}{3\alpha - 1} \left(\frac{1}{2} - \alpha \right) 2q.$$

²stated here in our context.

Now, let $\psi \in G_q$, *i.e.* ψ is a minimizer of I_q . By the Lagrange multipliers theory, there exists $\theta_q \in \mathbb{R}$ such that

$$(2.25) \quad D^\alpha \psi - \frac{1}{2} \psi^2 + \theta_q \psi = 0.$$

By using the energy and Pohojaev identities, we deduce exactly as in (2.20) and (2.21) that

$$(2.26) \quad \int_{\mathbb{R}} |D^{\frac{\alpha}{2}} \psi|^2 dx = \frac{\theta_q}{3\alpha - 1} \int_{\mathbb{R}} \psi^2 dx = \frac{2q\theta_q}{3\alpha - 1}$$

and

$$(2.27) \quad \int_{\mathbb{R}} \psi^3 dx = \frac{6\alpha\theta_q}{3\alpha - 1} \int_{\mathbb{R}} \psi^2 dx = \frac{12q\alpha\theta_q}{3\alpha - 1}.$$

Identities (2.26) and (2.27) imply in particular that $\theta_q > 0$ and $\int_{\mathbb{R}} \psi^3 dx > 0$, since $\alpha > \frac{1}{3}$.

Next, we prove that ψ must be positive. Indeed, recall that

$$|D^{\frac{\alpha}{2}}(|\psi|)|_2 \leq |D^{\frac{\alpha}{2}} \psi|_2,$$

for $\frac{1}{2} < \alpha < 1$. This claim follows for example from estimate (2.10) in [21]. Therefore, we deduce that $E(|\psi|) \leq E(\psi)$ and $M(|\psi|) = q$, since we also have

$$(2.28) \quad \int_{\mathbb{R}} \psi^3 dx = \left| \int_{\mathbb{R}} \psi^3 dx \right| \leq \int_{\mathbb{R}} |\psi|^3 dx.$$

Moreover, if ψ is not positive on \mathbb{R} , then the inequality in (2.28) is strict, so that $E(|\psi|) < E(\psi)$, which is a contradiction since $\psi \in G_q$.

We compute as above that

$$J^\alpha(\psi) = \frac{(2q)^{\frac{1}{2}} (3\alpha - 1)^{1 - \frac{1}{2\alpha}}}{\theta_q^{1 - \frac{1}{2\alpha}} 6\alpha}.$$

On the one hand, since $J^\alpha(\psi) \geq J^\alpha(Q_{c_\star})$, it follows from (2.22) and the definition of c_\star in (2.23) that

$$(2.29) \quad \theta_q \leq c_\star.$$

On the other hand, another simple computation gives that

$$E(\psi) = \frac{\theta_q}{3\alpha - 1} \left(\frac{1}{2} - \alpha \right) 2q.$$

Since $\psi \in G_q$, we have $E(\psi) \leq E(Q_{c_\star})$ which implies from (2.24) that

$$(2.30) \quad \theta_q \geq c_\star,$$

in the case $\alpha > \frac{1}{2}$. We conclude gathering (2.29) and (2.30) that

$$(2.31) \quad \theta_q = c_\star.$$

Therefore $J^\alpha(\psi) = J^\alpha(Q)$ and we conclude from the uniqueness result in Theorem 2.13 that $\psi = Q_{c_\star}(\cdot - y)$, for some $y \in \mathbb{R}$. \square

Finally, as a consequence of Theorem 2.10 and Lemma 2.12, we get the orbital stability of the ground states.

Theorem 2.14. *Let $\frac{1}{2} < \alpha < 1$, $c > 0$ and $Q_c = cQ(c^{\frac{1}{\alpha}}\cdot)$, where Q is the ground state solution of (2.16). For every $\epsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in H^s(\mathbb{R})$, $s > s_\alpha = \frac{3}{2} - \frac{3\alpha}{8}$, satisfy*

$$(2.32) \quad \|u_0 - Q_c\|_{\frac{\alpha}{2}} < \alpha,$$

then the corresponding solution u emanating from u_0 of (1.1) satisfies

$$(2.33) \quad \inf_{y \in \mathbb{R}} \|u(\cdot, t) - Q_c(\cdot + y)\|_{\frac{\alpha}{2}} < \epsilon$$

for all $t \in [0, T_s)$, where T_s is the maximal time of existence of u .

Remark 2.1. The (orbital) stability statement in Theorem 2.10 is a conditional one. It would become *unconditional* provided one establishes the *global* well-posedness of the Cauchy problem for data in the space $H^s(\mathbb{R})$, $s \leq \frac{\alpha}{2}$ when $\alpha > 1/2$. As previously mentioned, the best known result ([38]) establishes the local well-posedness of the Cauchy problem in $H^s(\mathbb{R})$, $s > \frac{3}{2} - \frac{3\alpha}{8}$, for any $\alpha > 0$. On the other hand it is proved in [25] that when $\alpha > \frac{1}{2}$, global weak L^2 solutions exist, as well as global $H^{\frac{\alpha}{2}}$ weak solutions, uniqueness being unknown. Also the numerical simulations of [34] suggest that no finite time blow-up occurs when $\alpha > \frac{1}{2}$, at least for smooth and localized initial data. Recall that when $1 < \alpha < 2$ the Cauchy problem is globally well-posed for initial data in $L^2(\mathbb{R})$ ([26]).

Remark 2.2. It has been established in [29] that the ground state is spectrally stable when $\alpha > \frac{1}{2}$.

Remark 2.3. The results above extend *mutatis mutandis* to the generalized fractional KdV equation

$$(2.34) \quad u_t + u^p u_x - D^\alpha u_x = 0, \quad u(\cdot, 0) = u_0$$

in the L^2 subcritical case, that is $\alpha > \frac{p}{2}$.

Remark 2.4. It would be interesting to prove the *asymptotic stability* of the ground states of (1.1) and also the existence (and stability) of *multisoliton solutions* of (1.1). Such solutions have been proven to exist and to be stable (in the subcritical case) for the generalized Korteweg-de Vries equations (see [44, 50]).

2.1. Remarks on instability. Instability of solitary wave solutions of the gKV equation

$$(2.35) \quad u_t + u_x + u^p u_x + u_{xxx} = 0,$$

has been established in [8] when $p > 4$ and in [45] for $p = 4$.

The mechanism of instability and the links with finite type blow-up are now well understood in the L^2 critical case $p = 4$ (see [46], [49, 48, 47] for theoretical studies and [33] for numerical simulations).

However a precise description of the instability in the super critical case $p > 4$ and in particular the proof of finite type blow-up are not known. Note that the link between instability and finite type blow-up is strongly suggested by the numerical simulations in [4] and [33] (where the L^2 critical case is also considered).

We now turn to the expected instability of the fKdV solitary waves when $\frac{1}{3} < \alpha \leq \frac{1}{2}$. The numerical simulations in [35] suggest that the instability mechanism is via finite time blow-up, similar to the KdV L^2 critical when $\alpha = 1/2$ and to the KdV L^2 supercritical case when $1/3 < \alpha < 1/2$. Proving such results appears to be out of reach, and we should restrict to the mere instability proof. As in [8] the first step is to give a sense to the formal conserved quantity

$$(2.36) \quad I(u) = \int_{\mathbb{R}} u dx.$$

Exactly as in Proposition 2.1 in [8], one checks that if $u_0 \in H^s(\mathbb{R})$, $s \geq 1 + \alpha$ is such that $\int_{-\infty}^{\infty} u_0(x) dx$ converges as a generalized Riemann integral, then $I(u(t))$ converges for any $t \in [0, T_s(u_0))$ and is constant, where $T_s(u_0)$ is the lifespan of the solution u of the corresponding Cauchy problem.

Again as in [8] one has to estimate how fast the tail of $I(u)$ near infinity grows with t . This cannot be deduce directly from Theorem 2.2 in [8] since

$$G_{\alpha}(x) = \int_{-\infty}^{\infty} e^{i(x\xi - \xi|\xi|^{\alpha})} d\xi$$

is not a bounded function of x when $\alpha < 1$.

Actually, (see [55]), $G_{\alpha}(x) = O(x^{-(\alpha+2)})$ as $x \rightarrow +\infty$ and oscillates when $x \rightarrow -\infty$, growing as $|x|^{(1-\alpha)/2\alpha}$.

In order to prove the equivalent of Theorem 2.2 in [8], one would need to impose a (one sided) decay property to u_0 insuring that the resulting solution of the Cauchy problem decays sufficiently to the left to compensate the growth of the fundamental solution.

3. THE FBBM EQUATION

As previously noticed an alternative to the toy model (1.1) is the fractional Benjamin-Bona-Mahony equation (fBBM) (1.2).

A solitary wave solution $u_c(x, t) = \phi(x - ct)$, $c > 0$ of (1.2) satisfies the equation

$$(3.1) \quad (c + D^{\alpha})u - \frac{u^2}{2} = 0.$$

Existence and stability issues for (3.1) have been considered in [60] when $\alpha > 1$ but the proofs therein extend readily to the case $\alpha < 1$.

More precisely, Zeng considers the minimization problem

$$(3.2) \quad I_q = \inf\{F(u) : u \in H^{\frac{\alpha}{2}}(\mathbb{R}) \text{ and } N(u) = q\},$$

where

$$F(u) = \int_{\mathbb{R}} (u^2 + |D^{\alpha/2}u|^2)$$

and

$$N(u) = \int_{\mathbb{R}} \left(\frac{u^2}{2} + \frac{u^3}{6} \right).$$

He thus considers the set of *ground state solutions* of (1.2), that is

$$G_q = \{u \in H^{\frac{\alpha}{2}}(\mathbb{R}) : N(u) = q \text{ and } F(u) = I_q\}.$$

The results established in [60] for $\alpha \geq 1$ and general nonlinearities $u^p u_x$ extends without any noticeable change in our case and imply the following theorem.

Theorem 3.1.

1. Assume that $\frac{1}{2} < \alpha < 1$. Then the set G_q is not empty and orbitally stable in $H^{\frac{\alpha}{2}}(\mathbb{R})$.
2. Assume that $\frac{1}{3} < \alpha < \frac{1}{2}$. Then there exists $q_0 = q_0(\alpha)$ such that for all $q > q_0$, G_q is not empty and orbitally stable in $H^{\frac{\alpha}{2}}(\mathbb{R})$.

Remark 3.1. 1. Again, the orbital stability results in Theorem 3.1 are conditional ones. A complete one would necessitate to prove a *global* well-posedness for the Cauchy problem associated to (1.2), when $\alpha > 1/3$. Due to the invariance of the $H^{\frac{\alpha}{2}}(\mathbb{R})$ norm, it would be sufficient to get a *local* well-posedness result in the same space. We recall that the best known result so far is given in [38] where local well-posedness is proven for initial data in $H^s(\mathbb{R})$, $s > \frac{3}{2} - \alpha$.

Note that the conservation of $E(u)$ implies by standard compactness methods the global existence of weak solutions in $H^{\frac{\alpha}{2}}(\mathbb{R})$, without uniqueness.

It is worth noticing that the numerical simulations in [34] suggest that a finite type blow-up may occur when $0 < \alpha \leq \frac{1}{3}$ but not when $\alpha > \frac{1}{3}$.

2. In the case of the generalized BBM equation (3.3), the critical value q_0 is associated to a critical velocity for the solitary waves, “fast” solitary waves are stable (see more details below). This fact relies strongly on the *explicit* formulas for the solitary waves. No such link seems to be known for fractional BBM equations.

As noticed in [7] for the generalized BBM equation

$$(3.3) \quad u_t + u_x + u^p u_x - u_{xxt} = 0,$$

the stability theory of solitary waves is “a little more complex” than for the corresponding generalized KdV equation (2.35) for which any solitary wave of *arbitrary* positive velocity is unstable when $p \geq 4$.

In fact (see [56]) solitary waves of (3.3) of arbitrary positive velocity are stable when $p < 4$ but when $p \geq 4$ there exists $c^* = c^*(p)$ such that the solitary waves of velocity $c < c^*$ are unstable while those of velocity $c > c^*$ are stable.

Furthermore the mechanism of instability is different since the Cauchy problem for gBBM is globally well posed in $H^1(\mathbb{R})$ for any p . The numerical simulations in [7] suggest that an unstable solitary wave will jump to a stable, faster one. No rigorous proof of this fact exists to our knowledge.

Instability results for generalized fBBM type equations are provided in [56] when $\alpha \geq 1$ in our notations. The proof does not extend easily to the case $\alpha < 1$ (they use properties of the multiplier $m(\xi) = 1 + |\xi|^\alpha$ that are no more valid when $\alpha < 1$).

4. REMARKS ON THE KP CASE

We consider now briefly the KP I version of (1.1), that is

$$(4.1) \quad u_t + uu_x - D_x^\alpha u_x + \epsilon \partial_x^{-1} u_{yy} = 0, \quad \text{in } \mathbb{R}^2 \times \mathbb{R}_+, \quad u(\cdot, 0) = u_0,$$

where $\epsilon = 1$ corresponds to the fKP II equation and $\epsilon = -1$ to the fKP I equation. Here D_x^α denotes the Riesz potential of order $-\alpha$ in the x direction, *i.e.* D_x^α is defined via Fourier transform by $(D_x^\alpha f)^\wedge(\xi, \eta) = |\xi|^\alpha \widehat{f}(\xi, \eta)$.

In addition to the L^2 norm, (4.1) conserves formally the energy (Hamiltonian)

$$(4.2) \quad H_\alpha(u) = \int_{\mathbb{R}^2} \left(\frac{1}{2} |D_x^{\frac{\alpha}{2}} u|^2 - \epsilon \frac{1}{2} |\partial_x^{-1} u_y|^2 - \frac{1}{6} u^3 \right).$$

The corresponding energy space is

$$Y_\alpha = \{u \in L^2(\mathbb{R}^2) : D_x^{\frac{\alpha}{2}} u, \partial_x^{-1} u_y \in L^2(\mathbb{R}^2)\}.$$

The first question is to which values of α correspond to the L^2 and the energy critical cases?

For the generalized KP-I equations

$$(4.3) \quad u_t + u^p u_x + u_{xxx} - \partial_x^{-1} u_{yy} = 0,$$

the corresponding values of p are respectively $p = 4/3$ and $p = 4$ (see [10, 11, 12]).

One checks readily that the transformation

$$u_\lambda(x, y, t) = \lambda^\alpha u(\lambda x, \lambda^{\frac{\alpha+2}{2}} y, \lambda^{\alpha+1} t)$$

leaves (4.1) invariant.

Moreover, $|u_\lambda|_2 = \lambda^{\frac{3\alpha-4}{4}} |u|_2$, so that $\alpha = \frac{4}{3}$ is the L^2 critical exponent.

The energy critical value of α is obviously related to the non existence of localized solitary waves. One has :

Proposition 4.1. *Assume that $0 < \alpha \leq \frac{4}{5}$ when $\epsilon = -1$ or that α is arbitrary when $\epsilon = 1$. Then (4.1) does not possess non trivial solitary waves in the space $Y_\alpha \cap L^3(\mathbb{R}^2)$.*

Proof. It is handy to write (4.1) as

$$(4.4) \quad \begin{cases} -cu_x + uu_x - D_x^\alpha u_x + \epsilon v_y = 0 \\ v_x = u_y, \end{cases}$$

Adapting the method in [10], we multiply successively the first equation by xu and yv . After some integrations by parts (which can be justified by a standard truncation in space procedure and a truncation of low frequencies as in [52]) one obtains the two identities:

$$(4.5) \quad \int_{\mathbb{R}^2} \left(\frac{c}{2}u^2 - \frac{1}{3}u^3 + \epsilon \frac{1}{2}v^2 + \frac{\alpha+1}{2} |D_x^{\frac{\alpha}{2}} u|^2 \right) = 0,$$

$$(4.6) \quad \int_{\mathbb{R}^2} \left(-\frac{c}{2}u^2 + \frac{1}{6}u^3 - \epsilon \frac{1}{2}v^2 - \frac{1}{2} |D_x^{\frac{\alpha}{2}} u|^2 \right) = 0.$$

On the other hand, the energy identity yields

$$(4.7) \quad \int_{\mathbb{R}^2} \left(-cu^2 + \frac{1}{2}u^3 + \epsilon v^2 - |D_x^{\frac{\alpha}{2}} u|^2 \right) = 0.$$

Subtracting (4.6) from (4.5) the cubic term from (4.5) yields

$$(4.8) \quad \int_{\mathbb{R}^2} \left(cu^2 - \frac{1}{2}u^3 + \epsilon v^2 + \frac{\alpha+2}{2} |D_x^{\frac{\alpha}{2}} u|^2 \right) = 0,$$

and adding with (4.7) we obtain

$$(4.9) \quad \int_{\mathbb{R}^2} \left(2\epsilon v^2 + \frac{\alpha}{2} |D_x^{\frac{\alpha}{2}} u|^2 \right) = 0,$$

proving that no solitary wave exists, whatever α in the defocusing case $\epsilon = 1$.

In the focusing, fKP I, case $\epsilon = -1$, we use (4.9) successively in (4.5) and (4.7) to get

$$(4.10) \quad \int_{\mathbb{R}^2} \left(\frac{c}{2}u^2 - \frac{1}{3}u^3 + \frac{3\alpha+4}{2\alpha}v^2 \right) = 0,$$

and

$$(4.11) \quad \int_{\mathbb{R}^2} \left(-cu^2 + \frac{1}{2}u^3 - \frac{\alpha+4}{\alpha}v^2 \right) = 0.$$

Eliminating v we obtain

$$(4.12) \quad \int_{\mathbb{R}^2} \left(c\alpha u^2 + \frac{4-5\alpha}{12}u^3 \right) = 0.$$

On the other hand, adding (4.5) and (4.6) yields

$$(4.13) \quad \frac{1}{3} \int_{\mathbb{R}^2} u^3 = \alpha \int_{\mathbb{R}^2} |D_x^{\frac{\alpha}{2}} u|^2,$$

which with (4.12) implies

$$(4.14) \quad \int_{\mathbb{R}^2} \left(cu^2 + \frac{4-5\alpha}{4} |D_x^{\frac{\alpha}{2}} u|^2 \right) = 0,$$

which proves that no solitary waves exist in this case when $\alpha \leq \frac{4}{5}$. \square

To go further it might be useful to consider the situation for the generalized KPI equation (4.3). Existence of solitary waves is established in [10] in the energy subcritical case $1 \leq p < 4$, by solving the variational problem I_λ consisting in minimizing the energy norm with the constraint

$$\int_{\mathbb{R}^2} u^{p+2} = \lambda.$$

To define the notion of *ground state* for (4.3), we introduce the energy

$$E_{KP}(\psi) = \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x \psi)^2 + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x^{-1} \partial_y \psi)^2 - \frac{1}{2(p+2)} \int_{\mathbb{R}^2} \psi^{p+2},$$

and we define the action

$$S(N) = E_{KP}(N) + \frac{c}{2} \int_{\mathbb{R}^2} N^2.$$

We term *ground state*, a solitary wave N which minimizes the action S among all finite energy non-constant solitary waves of speed c of (4.3) (see [10] for more details). It is proven in [10] that when $1 \leq p < 4$, the solutions of the minimization problem I_λ are ground states. Moreover (see [12]), when $1 \leq p < \frac{4}{3}$, the ground states are minimizers of the Hamiltonian E_{KP} with prescribed mass (L^2 norm). This implies (by an argument *à la Cazenave-Lions*) the orbital stability of the set of ground states (see also [43]). The *uniqueness*, up to the trivial symmetries of the ground states is a challenging open question.³ It is furthermore proven in [12] that any ground state (and in fact any cylindrically symmetric solitary wave) is unstable when $p > \frac{4}{3}$.

The instability result was improved by Liu [42] who used invariant sets of the generalized KP I flow together with the virial argument above to prove the existence of initial data leading to blow-up in finite time of $|u_y(.,t)|_2$ when $p \geq \frac{4}{3}$. This leads to a strong instability result (by finite time blow-up of $|u_y(.,t)|_2$) of the solitary waves when $2 < p < 4$.

³The stability result in [12] is a *conditional one* when $p \neq 1$ by lack of the *global* well-posedness of the corresponding Cauchy problem. Recall that the Cauchy problem for the KPI equation itself ($p = 1$) is globally well-posed in appropriate spaces! , including the energy space (see [53, 27]).

In order to check to what extent the above results could be extended to the fKP equation one has as a first step to establish a fractionary Gagliardo-Nirenberg inequality that allows for the Sobolev embedding of the energy space Y_α into $L^p(\mathbb{R}^2)$, $p \leq 3$.

The following inequality is a special case of Lemma 2.1 in [6] which considers only $1 \leq \alpha \leq 2$ but a close inspection at the proof reveals that it is still valid when $\frac{4}{5} < \alpha < 1$.

Lemma 4.2. *Let $\frac{4}{5} < \alpha < 1$. For any $f \in Y_\alpha$ one has*

$$|f|_3^3 \leq c |f|_2^{\frac{5\alpha-4}{\alpha+2}} \|f\|_{H_x^{\frac{\alpha}{2}}}^{\frac{18-5\alpha}{2(\alpha+2)}} |\partial_x^{-1} f_y|_2^{\frac{1}{2}},$$

where $\|\cdot\|_{H_x^{\frac{\alpha}{2}}}$ denotes the natural norm on the space

$$H_x^{\frac{\alpha}{2}}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : D_x^{\frac{\alpha}{2}} f \in L^2(\mathbb{R}^2)\}.$$

Lemma 4.2 implies obviously the embedding $Y_\alpha \hookrightarrow L^3(\mathbb{R}^2)$ if $\frac{4}{5} < \alpha < 1$ and is the starting point for an existence theory of solitary waves to fKPI equations which will be developed elsewhere [39]. Note that some results for the case $\alpha = 1$ (the KPI-Benjamin-Ono equation) are given in [20, 54].

Remark 4.1. Concerning the Cauchy problem for fKPI, one could conjecture a finite time blow-up of $|u_y|_0$ when $\frac{4}{5} < \alpha < \frac{4}{3}$ as Liu proved for the gKPI, explaining for instance the (expected) instability of KPI-BO ground states. We refer to a subsequent work [39] for a study of global *weak solutions* to fKP equations.

5. FINAL REMARKS

As already noticed, the precise description of the (expected) instability mechanism of the solitary waves of (1.1) when $\frac{1}{3} < \alpha \leq \frac{1}{2}$ seems out of reach for the moment. According to the numerical simulations in [34], the instability seems to be due to blow-up. Recall that this issue is still open for the generalized KdV equation (that is (2.34) with $\alpha = 2$) when $p > 4$, the critical case $p = 4$ being treated in [46].

Similarly, the description of the (expected) instability of slow solitary waves of the fBBM equation when $\frac{1}{3} < \alpha \leq \frac{1}{2}$ is not known. Recall that a corresponding rigorous description of solitary waves of the gBBM when $p > 4$ is still an open problem.

On the other hand, the computations in [35] seem to indicate that the *soliton resolution conjecture* (see [57]) is true for both the fKdV and fBBM equations in the stable range $\frac{1}{2} < \alpha \leq 1$.

Acknowledgements. The Authors were partially supported by the Brazilian-French program in mathematics. J.-C. S. acknowledges support from the

project ANR-GEODISP of the Agence Nationale de la Recherche. F.L and D.P. were partially supported by CNPq and FAPERJ/Brazil.

REFERENCES

1. J.P. ALBERT, *Concentration Compactness and the stability of Solitary-Wave Solutions to Nonlocal Equations*, Contemporary Mathematics **221** (1999), 1-29.
2. J. ALBERT, J. L. BONA AND J.-C. SAUT, *Model equations for waves in stratified fluids*, Proc. Royal Soc. London A, **453**, (1997), 1233-1260.
3. O. E. BESOV, V.P. ILIN AND S.M. NIKOLS'SKII, *Integral representations of functions and imbedding theorems*, Volume I, John Wiley and Sons, New York, 1978.
4. J.L. BONA, V.A. DOUGALIS, O.A. KARAKASHIAN AND W.R. MCKINNEY, *Conservative, high-order numerical schemes for the generalized Korteweg-de Vries equation*, Philos. Trans. Roy. Soc. London Ser. A **351**, 1695 (1995), 107-164.
5. J.L.BONA AND H. KALISH, *Singularity formation in the generalized Benjamin-Ono equation*, Discrete Cont. Dyn. Systems **11** (1) (2004), 27-45.
6. J.L.BONA, YUE LIU AND M.M. TOM, *The Cauchy Problem and Stability of Solitary-Wave Solutions for RLW-KP-Type Equations*, J. Diff. Eq. **185** (2002), 437-482.
7. J.L. BONA, W.R. MCKINNEY AND J.M. RESTREPO, *Stable and unstable solitary -wave solutions of the generalized long -wave equation*, J. Nonlinear Sci. **10** (2000), 603-608.
8. J. L. BONA, P.E. SOUGANIDIS AND W.A. STRAUSS, *Stability and instability of solitary waves of the Korteweg- de Vries equations*, Proc. Roy. Soc. London A **411** (1987), 395-412.
9. T.B. BENJAMIN, *The stability of solitary waves*, Proc. R. Soc. Lond. **328** (1972), 153-183.
10. A. DE BOUARD AND J.-C. SAUT, *Solitary waves of generalized KP equations*, Annales IHP Analyse non Linéaire, **14** (2), (1997) 211-236.
11. A. DE BOUARD AND J.-C. SAUT, *Symmetries and decay of the generalized KP solitary waves*, SIAM J. Math. Anal., **28** (5) (1997), 1064-1085.
12. A. DE BOUARD AND J.-C. SAUT, *Remarks on the stability of generalized KP solitary waves*, in Mathematical Problems in the Theory of Water Waves, F. Dias, J.-M. Ghidaglia, J.-C. Saut (Editors), Contemporary Mathematics **200**, AMS (1996), 75-84.
13. J.C. BRONSKY AND V. M. HUR, *Modulational instability and variational structure*, arXiv : 1303.2605 (2013).
14. T. CAZENAVE AND P.-L. LIONS, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Commun. Math. Phys. **85** (1982), 549-561.
15. J. CHEN, B. GUO AND Y. HAN, *Blow-up and instability of solitary wave solutions to a generalized Kadomtsev-Petviashvili equation and two-dimensional Benjamin-Ono equation*, Proc. Roy. Soc A **464** (2008), 49-64.
16. Y. CHO, H. HAJAJEJ, G. HWANG AND T. OZAWA, *On the orbital stability of fractional Schrödinger equations*, Comm. Pure Appl. Anal. **13** (3) (2014), 1267-1282.
17. M. EHRNSTRÖM, M.D. GROVES AND E. WAHLÉN, *On the existence and stability of solitary-wave solutions to a class of evolution equations of Whitham type*, Nonlinearity **25** (2012), 2903-2936.
18. K. EL DIKA, *Asymptotic stability of solitary waves for the Benjamin-Bona-Mahony equation*, Discrete Contin. Dyn. Syst. **13** (3) (2005), 58-3-622.
19. K. EL DIKA AND Y. MARTEL, *Stability of N-solitary waves for the generalized BBM equation*, Dyn. Partial Equ. **1** (4) (2004), 401-437.
20. A. ESFAHANI, *Remarks on solitary waves of the generalized two-dimensional Benjamin-Ono equation*, Applied Math. and Comp. **218** (2011), 308-323.
21. P. FELMER, A. QUAAS AND J. TAN, *Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian*, Proc. Roy. Soc. Edinburgh, **142** A, (2012), 1237-1262.
22. R.L. FRANK, *On the uniqueness of ground states of non-local equations*, arXiv:1109.4049v1, 19 Sep 2011.
23. R.L. FRANK, E. LENZMANN AND L. SILVESTRE, *Uniqueness of radial solutions for the fractional Laplacian*, arXiv:1302.2652v1 [math.AP] 11 Feb 2013.
24. R.L. FRANK AND E. LENZMANN, *On the uniqueness and non-degeneracy of ground states of $(-\Delta)^s Q + Q - Q^{\alpha+1} = 0$ in \mathbb{R}* , Acta Math. **210** (2) (2013), 261-318.
25. J. GINIBRE AND G. VELO, *Smoothing Properties and Existence of Solutions for the Generalized Benjamin-Ono Equations*, J. Diff. Eq. **93** (1991), 150-212.

26. S. HERR, A. IONESCU, C. E. KENIG AND H. KOCH, *A para-differential renormalization technique for nonlinear dispersive equations*, Comm. Partial Diff. Eq., **35** (2010), no. 10, 1827–1875.
27. A. D. IONESCU, C. KENIG AND D. TATARU, *Global well-posedness of the initial value problem for the KP I equation in the energy space*, Invent. Math. **173** 2 (2008), 265–304.
28. M.A. JOHNSON, *Stability of small periodic waves in fractional KdV type equations*,
29. T. KAPITULA AND A. STEFANOV, *A Hamiltonian-Krein (instability) index theory for KdV-like eigenvalue problems*, Studies Appl. Math. **132** (2014), 183–211.
30. C.E. KENIG, AND Y. MARTEL, *Asymptotic stability of solitons for the Benjamin-Ono equation*, Revista Mat. Iberoamericana **25** (2009), 909–970.
31. C.E. KENIG, Y. MARTEL AND L. ROBBIANO, *Local well-posedness and blow-up in the energy space for a class of L^2 critical dispersion generalized Benjamin-Ono equations*, Ann. I. H. Poincaré, **28** (2011), 853–887.
32. C. E. KENIG, G. PONCE AND L. VEGA, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle*, Comm. Pure Appl. Math., **46** (1993), 527–620.
33. C. KLEIN AND R. PETER *Numerical study of blow-up in solutions to generalized Korteweg-de Vries equations*, arXiv:1307.0603v1 [math-ph] 2 Jul 2013.
34. C.KLEIN AND J.-C.SAUT, *A numerical approach to blow-up issues for dispersive perturbations of Burgers equation*, Physica D, **295-296** (2015), 46–65.
35. C.KLEIN AND J.-C.SAUT, *IST versus PDE, a comparative study*, arXiv:1409.2020v2 [math.AP] 9 Sep 2014.
36. D. LANNES, *Water waves: mathematical theory and asymptotics*, Mathematical Surveys and Monographs, 188. American Mathematical Society, Providence, RI, 2013.
37. D. LANNES AND J.-C. SAUT, *Remarks on the full dispersion Kadomtsev-Petviashvili equation*, Kinetic and Related Models, American Institute of Mathematical Sciences **6**, Number 4 (2013), 989–1009.
38. F. LINARES, D. PILOD AND J.-C. SAUT, *Dispersive perturbations of Burgers and hyperbolic equations I : local theory*, SIAM J. Math. Anal., **46** (2) (2014), 1505–1537.
39. F. LINARES, D. PILOD AND J.-C. SAUT, *Remarks on the fractionary Kadomtsev-Petviashvili equations*, In preparation.
40. P.-L. LIONS, *The concentration compactness principle in the calculus of variations. The locally compact case, part 1*, Ann. Inst. H. Poincaré **1** (1984), 109–145.
41. P.-L. LIONS, *The concentration compactness principle in the calculus of variations. The locally compact case, part 2*, Ann. Inst. H. Poincaré **4** (1984), 223–283.
42. YUE LIU, *Blow-up and instability of solitary-wave solutions to a generalized Kadomtsev-Petviashvili equation*, TAMS **353** (2001), 191–208.
43. YUE LIU AND X. P. WANG, *Nonlinear stability of solitary waves of generalized Kadomtsev-Petviashvili equations*, Comm. Math. Phys. **183**, no. 2 (1997), 253–266.
44. Y. MARTEL, *Asymptotic N-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equation*, Amer. J. Math. **127** (5) (2005), 1103–1140.
45. Y. MARTEL AND F. MERLE, *Instability of solitons for the critical generalized Korteweg-de Vries equation*, GAFA **11** (2001), 74–123.
46. Y. MARTEL AND F. MERLE, *Blow up in finite time and dynamics of blow up solutions for the critical generalized KdV equation*, J. Amer. Math. Soc. **15** (2002), 617–664.
47. Y. MARTEL, F. MERLE AND P. RAPHAËL, *Blow up for the critical gKdV equation III: exotic regimes*, submitted.
48. Y. MARTEL, F. MERLE AND P. RAPHAËL, *Blow up for the critical gKdV equation II: minimal mass blow up*, submitted
49. Y. MARTEL, F. MERLE AND P. RAPHAËL, *Blow up for the critical gKdV equation I: dynamics near the solitary wave*
50. Y. MARTEL, F. MERLE AND T-P. TSAI, *Stability and asymptotic stability of the sum of N solitons for subcritical gKdV equations*, Comm. Math. Phys. **231** (2) (2002), 347–379.
51. T. MIZUMACHI, *Asymptotic stability of solitary wave solutions to the regularized long-wave equation*, J. Differ. Equ. **200** (2004), 312–341.
52. L. MOLINET, *On the asymptotic behavior of solutions to the (generalized) Kadomtsev-Petviashvili-Burgers equation*, J. Diff. Eq. **152** (1999), 30–74.

53. L. MOLINET, J.C. SAUT AND N. TZVETKOV, *Global well-posedness for the KP-I equation*, Math. Annalen 324, (2002), pp. 255-275. Correction : Math. Ann. 328 (2004), 707–710.
54. G. PRECIADO LOPES AND F.H. SORIANO MÉNDEZ, *On the existence and analyticity of solitary wave solutions to a two-dimensional Benjamin-Ono equation*, preprint 2013.
55. A. SIDI, C. SULEM AND P.-L. SULEM, *On the long time behavior of a generalized KdV equation*, Acta Applicandae Mathematicae **7**, (1986), 35-47.
56. P.E. SOUGANIDIS AND W.A. STRAUSS, *Instability for a class of dispersive solitary waves*, Proc. Roy. Soc. Edinburgh **114A** (1990), 195-212.
57. T. TAO, *Why are solitons stable?*, Bull. AMS **46** (1) (2009), 1–33.
58. M.I WEINSTEIN, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math.Phys. **87** (1983), 567-576.
59. G.B. WHITHAM, *Linear and nonlinear waves*, Wiley, New York 1974.
60. LEI ZHENG, *Existence and stability of solitary wave solutions of equations of Benjamin-Bona-Mahony type*, J. Differential Equations **188** (2003), 1-32.

IMPA, ESTRADA DONA CASTORINA 110, RIO DE JANEIRO 22460-320, RJ BRASIL
E-mail address: `linares@impa.br`

INSTITUTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, CAIXA POSTAL 68530 CEP 21941-97, RIO DE JANEIRO, RJ BRASIL
E-mail address: `didier@im.ufrj.br`

LABORATOIRE DE MATHÉMATIQUES, UMR 8628,, UNIVERSITÉ PARIS-SUD ET CNRS,, 91405 ORSAY, FRANCE
E-mail address: `jean-claude.saut@math.u-psud.fr`